# Lagrange Multipliers 1

Suppose that we want to find the maximum or minimum of a function of three variables $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$. One possible strategy is to convert the problem into a problem involving only two variables by solving the equation $g(x, y, z) = 0$ for one of the variables in terms of the other two and then substituting this into the function $f$ to convert it into a function of two variables.

**Example**

In this example we want to maximize the volume of a box subject to the condition that we want the surface area to be 24. Translated into mathematical language, this means that we want to maximize the function

$$f(x, y, z) = xyz \quad \text{(length} \times \text{width} \times \text{height)}$$

subject to the constraint

$$g(x, y, z) = 2xy + 2xz + 2yz - 24 = 0.$$

What we could do here is to solve the constraint equation to obtain

$$z = \frac{12 - xy}{x + y}$$

and then substitute this into $f$ to get the function

$$f(x, y) = \frac{xy(12 - xy)}{x + y},$$

which we can then attempt to maximize using the techiniques of calculus of two variables.

However there are a couple of problems with this approach. Firstly, even with this simple example, the calculus is quite difficult, and secondly we may be dealing with a constraint that cannot even be solved for one variable in terms of the others.

To get around these problems we use the method of **Lagrange Multipliers**. This involves solving the following **Lagrange equations**:

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}, \quad \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}, \quad g = 0.$$

In the case of our box these are

$$yz = \lambda(2y + 2z), \quad xz = \lambda(2x + 2z), \quad xy = \lambda(2x + 2y), \quad 2xy + 2xz + 2yz - 24 = 0. \tag{1}$$

We now have to solve these and it is this step that is usually the most difficult one when using Lagrange multipliers. The problem is that the equations we end up with are in general non-linear and there is no algorithm, such as Gaussian ellimination, that we can use. This is why it is important to tackle a variety of different problems, so that you end up with a range of techniques ‘up your sleeve’. In this particular case a good strategy is to multiply the first equation in (1) by $x$, the second equation in (1) by $y$ and the third equation in (1) by $z$, since in this way the left hand sides of all these equations becomes $xyz$ and we can then equate the right hand sides. On doing this, the first three equations in (1) become

$$xyz = \lambda(2xy + 2xz), \quad xyz = \lambda(2xy + 2yz), \quad xyz = \lambda(2xz + 2yz),$$

so that

$$\lambda(2xy + 2xz) = \lambda(2xy + 2yz) = \lambda(2xz + 2yz). \tag{2}$$

We now have to consider two cases, depending on whether or not $\lambda$ is zero.

If $\lambda = 0$, then $yz = xz = xy = 0$ follows from the first three equations in (1), so that

$$2xy + 2xz + 2yz - 24 = 0 - 24 = -24$$

and the constraint equation is not satisfied. Thus there are no solutions in this case.

Note that this sort of careful argument is a feature of Lagrange multiplier problems and you have to make sure that you cover all cases to ensure that you don't miss any solutions. In particular it is vital to ensure that if you divide by some expression then it is not possible for it to be zero. If it is possible then you will have to consider this possibility separately otherwise you will omit some solutions.

We will now consider the case $\lambda \neq 0$. Since $\lambda \neq 0$, we can divide by $\lambda$ in (2) to obtain

$$2xy + 2xz = 2xy + 2yz = 2xz + 2yz.$$

Since we may assume that $x$, $y$ and $z$ are non -zero (for otherwise the volume of the box would be zero),

$$2xy + 2xz = 2xy + 2yz \text{ implies that } x = y.$$

Similarly

$$2xy + 2yz = 2xz + 2yz \text{ implies that } y = z.$$

Thus we have that $x = y = z$ and substituting this in the constraint equation we have

$$6x^2 = 24, \text{ so that } x = \pm 2.$$

Of course the solution $x = -2$ has no meaning in this problem (although it may have in others), so we can discount it.

Hence we have our final solution $x = y = z = 2$, giving a volume of 8.

This is not quite the end of our work though. The fact that these values are a solution to the Lagrange equations does not mean that a maximum or minimum necessarily occurs at this point. All is not lost however; what is true is that if a maximum or minimum occurs at a point then the Lagrange equations have to be satisfied there. So a general technique is to find all the solutions of the Lagrange equations that make sense in the particular problem being considered (for example in this problem we discarded solutions with negative or zero $x$, $y$ and $z$ as either making no sense or clearly not yielding a maximum) and then decide using other arguments whether what we are left with are maxima or minima or neither. The amount of detail you have to give in this step will depend on whether you are studying an introductory first year course or a more advanced one - your lecturer will be able to advise you on this. For our box problem, it is true that the solution at $x = y = z = 2$ is indeed a maximium but to show this rigourously takes more effort than is appropriate to include here.

Note that in this example we did not need to find the value of $\lambda$ (called the **Lagrange multiplier**) and this is often the case, although, depending on the problem, it may be better to find $\lambda$ on the way to finding $x$, $y$ and $z$. In other problems, mainly connected with economics, $\lambda$ has a concrete meaning and it enables us to find the solution to a related optimization problem without having to go to the effort of solving a new set of equations. An example of this sort of problem is given in **Lagrange Multipliers 2.**